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Efficiency of cropping system designs via dual designs*

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SUMMARY

The topic of the paper is the efficiency of unequally replicated crop rotation designs. Incorporating the model of incomplete block designs we deal with designs of the class GCIB_2. The extended notion of the canonical efficiency factors and average efficiency factor for unequally replicated designs are applied. The efficiency factors are derived via dual designs. Considerable simplicity is gained by the appropriate relations concerning parental and dual design. The key property exploited is that dual designs are cyclic designs.

KEY WORDS: cyclic design, crop rotation, dual design, efficiency, generalized cyclic design.

1. Preliminaries

We start with a brief outline of the paper. In Section 1 we state the problem and establish conventions for notation. In Section 2 we show that for the efficiency related purposes, crop rotation designs may be regarded as the designs having patterns admitting some simplified methods of the study. Sections 3 and 4 provide some familiar results pertaining to the efficiency of unequally replicated designs and relationships concerning duality and efficiency. Finally, in Section 4.1 we derive some simplification of the study depending on the pattern of a matrix attached to the dual design. In Section 5 we conclude our efficiency investigations with some examples. Appendix furnishes the references concerning algebra of circulant matrices.

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1.1. Crop rotation

Crop rotation (also referred to as *cropping system)* is the fixed sequence of species of plants which appear on the same plot in the course of many years. We start an experiment with all plants included into the sequence. Two examples of four-field cropping systems are given below.

year	plot no.			
no.	1	2	3	4
1	wheat	beans	wheat	wheat
2	wheat	wheat	beans	wheat
3	wheat	wheat	wheat	beans
4	beans	wheat	wheat	wheat

year	plot no.			
no.	1	2	3	4
1	wheat	potato	wheat	lupin
2	lupin	wheat	potato	wheat
3	wheat	lupin	wheat	potato
4	potato	wheat	lupin	wheat

The key assumption in our investigations is that different sequences of plants create various levels of soil fertility in the plots. We treat such additional effects of soil fertility accumulated during the full rotation as the *treatment effects*. From the agricultural point of view the usefulness of rotation should be appraised on the basis of the yields of all species. Because of the character of the plant products (for example a field of corn and seeds of rape) such the yields are incomparable. It is often in practice that the same species appears in the rotation more than once, therefore we can treat such a plant as the *test crop* and we only take the yields of this species for the statistical analysis. In the crop rotations considered wheat is treated as the test crop. In other words, the wheat yields are regarded as the treatment responses, i.e., responses to the different sequences of plants assigned to the respective plots.

1.2. Linear models

The usual linear model for crop rotation experiment (cf. Przybysz, 1982) specifies the observation as follows

$$y_{ijk} = \mu + \rho_i + \alpha_j + e_{ij} + \beta_k + \varphi_{jk} + \varepsilon_{ijk} ,$$

where μ is a general level effect, ρ_i, α_j, β_k represent the effects due to i-th replication ($i = 1, ..., r$), j-th plot ($j = 1, ..., v$) and the k-th year ($k = 1, ..., b$). Besides, e_{ij} and ε_{ijk} are random errors of experimental units and random technical errors, respectively, and φ_{jk} stands for interaction component.

The model of the experiment can be written in matrix notation as

$$\mathbf{y} = \mathbf{1}\mu + \mathbf{R}'\rho + \mathbf{\Delta}'\alpha + \mathbf{e} + \mathbf{D}'\beta + \mathbf{G}'\varphi + \varepsilon .$$

It should be emphasized that such a design is not complete with respect to blocks (i.e. the plot responses are obtained not in every year). This fact is exploited in the further study. Since the binary matrices $\mathbf{\Delta}$ and $\mathbf{D}$ take a prominent place in our efficiency investigations, by means of them we can describe *the model of one replication* as follows

$$\mathbf{y}_* = \mathbf{\Delta}'_*\alpha + \mathbf{D}'_*\beta + \mathbf{e}_* ,$$

where $\mathbf{y}_*$ is $n \times 1$ vector of observations, $n = bk$, $\mathbf{\Delta}'_*, \mathbf{D}'_*$ are $n \times v$, $n \times b$ design matrices for treatments and years, α and β are $v \times 1$, $b \times 1$ vectors of treatment parameters and year effects, $\mathbf{e}$ is $n \times 1$ random vector of errors. Let us notice that the transformation led to a model of incomplete block design. For the remainder of the paper, the study will be concerned with this model.

Our interest focuses on supplemented designs, i.e., designs comprising two competing cropping systems. In general, the incidence matrix of such a design admits the representation (see Nigam et al., 1988)

$$\mathbf{N}^* = \begin{pmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \end{pmatrix} .$$

The incidence submatrices stay in the close relation with the patterns of allocation of plants to the plots in the respective rotations (cf. the tables in the previous section). Namely, the rows are related to the plots, the columns correspond to the years, "one" indicates that wheat was on the plot, zero refers to the other species. Because of the cyclical set-up of the crop rotation experiment, the treatment replications stay invariant throughout each of the competing systems, and are equal to r_1 and r_2, say.

2. Generalized cyclic designs

In a class of generalized cyclic incomplete blocks, henceforth referred to as GCIB_m, the v treatments are divided into m groups with n elements in each group (cf. John, 1987). A design is obtained by successive addition of m, modulo v, to the elements of one or more initial blocks. Each initial block contributes n blocks to the design. It follows that in such a design the treatment replications are invariant throughout each residue class, where the i-th residue class consists of treatments labelled by the following numbers

$$i, i+m, \ldots, i+m(n-1), \qquad i = 0, \ldots, m-1 \ .$$

2.1. Cropping system designs as $GCIB_2$

The supplemented designs introduced in the previous section are covered by the class of generalized cyclic designs. Namely, by a suitable rearrangement (renumbering) of treatments, the supplemented design can be formulated as GCIB_2. Let us consider incidence matrix of the supplemented design comprising the cropping systems introduced in the Section 1.

$$\mathbf{N}^* = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} .$$

By appropriate permutation of treatments (rows of incidence matrix) we obtain

$$\mathbf{N}_p^* = \begin{pmatrix} 1 & 0 & 1 & 1 \\ \mathbf{1} & 0 & \mathbf{1} & 0 \\ 1 & 1 & 0 & 1 \\ 0 & \mathbf{1} & 0 & \mathbf{1} \\ 1 & 1 & 1 & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & 1 & 1 & 1 \\ 0 & \mathbf{1} & 0 & \mathbf{1} \end{pmatrix} .$$

It is readily apparent that the initial block of such design contributes to remaining blocks by the cyclical addition of 2, modulo 8, to its elements. In a similar fashion it can be showed that the class of $GCIB_2$ includes the remaining supplemented designs that are of our interest.

Due to above characterization result in a structure of cropping system designs, we can tackle the problem of the efficiency according to the results by Jarrett et al. (1978). Because of the cyclical pattern of constructing, the subject of the generalized inverse and the pairwise efficiency of the $GCIB_m$ can be considered in a unified way. But these aspects are beyond the scope of our discussion, we only point out the applicable methods.

3. Efficiency of unequally replicated designs

Following Ceranka and Mejza (1979) the measure of the efficiency can be extended to the designs with unequal replications. Regarding the replication vector as fixed we can consider the $\mathbf{r}^{\delta}$-*basic contrasts* as the eigenvalues of the information matrix $\mathbf{C}$ with respect to the matrix $\mathbf{r}^{\delta}$. In other words, extending notion of the efficiency to the unequal replicate case, canonical efficiency factors e_i are obtained as the eigenvalues of the following symmetric matrix

$$\mathbf{C}^* = \mathbf{r}^{-\frac{\delta}{2}}\mathbf{C}\,\mathbf{r}^{-\frac{\delta}{2}} = \mathbf{I} - \mathbf{r}^{-\frac{\delta}{2}}\mathbf{N}\mathbf{k}^{-\delta}\mathbf{N}'\mathbf{r}^{-\frac{\delta}{2}} . \tag{1}$$

Then the average efficiency factor E is given by

$$E = \frac{v-1}{\sum_{i=1}^{v-1} e_i^{-1}} . \tag{2}$$

4. Dual block design

We aim at deriving the efficiency factors via dual designs. The dual design of a block design is obtained by interchanging the treatments and blocks in the parental design. Consequently, the incidence matrix of the dual design is $\mathbf{N}_d = \mathbf{N}'$. By (1) the average efficiency factor of the dual design is the harmonic mean of the eigenvalues of the following matrix

$$\mathbf{C}_d^* = \mathbf{k}^{-\frac{\delta}{2}}\mathbf{C}_d\,\mathbf{k}^{-\frac{\delta}{2}} = \mathbf{I}_b - \mathbf{k}^{-\frac{\delta}{2}}\mathbf{N}'\mathbf{r}^{-\delta}\mathbf{N}\,\mathbf{k}^{-\frac{\delta}{2}} . \tag{3}$$

Let us consider the case $b < v$. Let matrix $\mathbf{C}_d^*$ has the following non-zero eigenvalues $e_1^d, e_2^d, ..., e_{b-1}^d$. Then following (2) we can write

$$E_d = \frac{b-1}{\sum_{i=1}^{b-1} (e_i^d)^{-1}} .$$

Of the many relationships between the original design and dual one we make use of the following concerning the efficiency. Namely, the set of eigenvalues of the matrix $\mathbf{C}^*$ consists of those $(b-1)$ eigenvalues of the matrix $\mathbf{C}_d^*$ and $(v-b)$ eigenvalues equal to one (cf. John, 1987). Hence the average efficiency factor of the parental design is of the following form

$$E = \frac{v-1}{(v-b)+\sum_{i=1}^{b-1}(e_i^d)^{-1}} = \frac{v-1}{(v-b)+(b-1)E_d^{-1}} . \qquad (4)$$

4.1. Dual design of cropping system design

These results are very useful in the case of cropping systems designs. In the special case of the cropping systems with equal rotation lengths, i.e., for $b = v/2$, the advantage of the implementation of dual design is readily apparent. But the relationship given by (4) is also useful in other ways.

Namely, in the case of the cropping system designs we have $\mathbf{k}^{-\frac{\delta}{2}} = k^{-\frac{1}{2}}\,\mathbf{I}_b$, so we can write

$$\mathbf{C}_d^{*} = \mathbf{k}^{-\frac{\delta}{2}}\mathbf{C}_d\,\mathbf{k}^{-\frac{\delta}{2}} = \mathbf{I}_b - 1/k\,\mathbf{N}'\mathbf{r}^{-\delta}\mathbf{N} .$$

There have been some studies on characterization of the relationships between $GCIB_2$ and cyclic designs. The key references are John (1987, 1980). The second paper concerns the equally replicated set-up. We will not pursue the discussion on these results. Instead, we make a brief study of a structure of the matrix $\mathbf{C}_d^{*}$ in relation to the incidence matrix. Towards this, let us recall that the incidence matrix of cropping system design is block circulant matrix

$$\mathbf{N} = \begin{pmatrix} \mathbf{Z}_1 \\ \cdots \\ \mathbf{Z}_2 \end{pmatrix} ,$$

where $\mathbf{Z}_1$ and $\mathbf{Z}_2$ have properties of being circulant (see Appendix).

Let us consider the following product

$$\begin{aligned} \mathbf{N}'\mathbf{r}^{-\delta}\mathbf{N} &= \begin{pmatrix} \mathbf{Z}_1' & \vdots & \mathbf{Z}_2' \end{pmatrix} \begin{pmatrix} r_1^{-1}\mathbf{I} & 0 \\ 0 & r_2^{-1}\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{Z}_1 \\ \cdots \\ \mathbf{Z}_2 \end{pmatrix} \\ &= r_1^{-1}\mathbf{Z}_1'\mathbf{Z}_1 + r_2^{-1}\mathbf{Z}_2'\mathbf{Z}_2 . \end{aligned}$$

Since matrices $\mathbf{Z}_1$, $\mathbf{Z}_2$ admit the representation (6), therefore we conclude that the product mentioned above gives the circulant matrix. This statement can be justified following the relations (7). To trace the details of finding explicit formulae for coefficients c_i we refer to Appendix. The relation we have just established is very attractive from the computational point of view. Namely, the matrix $\mathbf{C}_d^{*}$ attached to the dual design is the circulant matrix, i.e., $\mathbf{C}_d^*$ admits the general representation

$$\mathbf{C}_d^* = \begin{pmatrix} c_0 & c_1 & \cdots & c_{b-2} & c_{b-1} \\ c_{b-1} & c_0 & & c_{b-3} & c_{b-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_2 & c_3 & \cdots & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{b-1} & c_0 \end{pmatrix} .$$

The advantage of the result is the simplicity of the evaluating procedure. Namely, a simpler formula becomes available by the following statement concerning circulant matrices (cf. John, 1987). The eigenvalues of the circulant matrix $\mathbf{C}_d^*$ can be obtained by the formula

$$e_i^d = \sum_{h=0}^{b-1} c_h \cos(2\pi i h \,/\, b) \;, \quad i = 0, 1, ..., b-1 \;. \tag{5}$$

This result allows us to conclude that the only calculations required are that implied by the relation (5).

5. Application

Our main objective is to derive the efficiency factors for a class of supplemented designs comprising four-field cropping systems. The relations (5) and (4) provide the meaningful simplification for our efficiency investigations. In our considerations, i.e., in the case of designs with equal rotation lengths, the dimension of the matrix $\mathbf{C}^*$ decreases by half owing to the dual design. It remains to compute $\mathbf{C}_d^*$ according to (3) and exploit the properties (5) and (4).

The results obtained are arranged in Table 1. The supplemented designs are signified by percent frequency of wheat in the competitive crop rotations. In other words, entries of the first column denote percentage frequency of "ones" in initial rows of respective circulants in composite incidence matrix $\mathbf{N}$. Subscript 2 concerns the cropping system in that wheat appears on the plot twice, year by year, subscript 1 refers to the remaining case (see, e.g., illustrative rotation in Section 1).

Table 1. Efficiency of supplemented designs

design	dual efficiency	efficiency
$\mathbf{50_2{-}25}$	.400	.609
$\mathbf{50_2{-}50_1}$	.643	.808
$\mathbf{75-25}$	.667	.824
$\mathbf{75-50_1}$	.747	.873
$\mathbf{75-50_2}$	.790	.898

6. Appendix

The *circulant matrix* $\mathbf{Z}$ can be written as

$$\mathbf{Z} = \sum_{h=0}^{v-1} a_h \mathbf{\Gamma}_h \ , \tag{6}$$

where $\mathbf{\Gamma}_h$ is $v \times v$ is *basic circulant matrix* whose initial row has 1 in the $(h+1)$-th column and zero elsewhere. Remaining rows are obtained by the circulant rotation of the initial row. Besides, it is easy to verify that

$$\mathbf{\Gamma}_i^{'} = \mathbf{\Gamma}_{v-i}, \quad \mathbf{\Gamma}_i \mathbf{\Gamma}_j = \mathbf{\Gamma}_{(i+j) \bmod v}. \tag{7}$$

It follows then that

$$\begin{aligned}
\mathbf{Z}'\mathbf{Z} &= \left(\sum_{i=0}^{v-1} a_i \mathbf{\Gamma}_i^{'}\right)\left(\sum_{j=0}^{v-1} a_j \mathbf{\Gamma}_j\right) = \sum_{i=0}^{v-1}\sum_{j=0}^{v-1} a_i a_j \Gamma_{v-i}\Gamma_j \\
&= \sum_{i=0}^{v-1}\sum_{j=i}^{v-1} a_i a_j \Gamma_{j-i} + \sum_{i=1}^{v-1}\sum_{j=0}^{i-1} a_i a_j \Gamma_{v-i+j} \\
&= \sum_{i=0}^{v-1}\sum_{k=0}^{v-i-1} a_i a_{k+i} \Gamma_k + \sum_{i=1}^{v-1}\sum_{k=v-i}^{v-1} a_i a_{k+i-v} \Gamma_k \\
&= \sum_{k=0}^{v-1}\sum_{i=0}^{v-k-1} \Gamma_k \ a_i a_{k+i} + \sum_{k=1}^{v-1}\sum_{i=v-k}^{v-1} \Gamma_k \ a_i a_{k+i-v} \\
&= \sum_{k=0}^{v-1} \Gamma_k \sum_{i=0}^{v-1} a_i a_{(k+i) \bmod v} \ .
\end{aligned}$$

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Efektywność układów płodozmianowych poprzez układy dualne

STRESZCZENIE

Praca dotyczy efektywności układów płodozmianowych. Układy tego typu są układami rozszerzonymi o nierównej liczbie replikacji i należą do szerokiej klasy uogólnionych układów cyklicznych $GCIB_2$. Do wyznaczenia współczynników efektywności wykorzystane są układy dualne, a dokładniej zależności pomiędzy ich efektywnością a efektywnością układu badanego. Cykliczny charakter macierzy incydencji oraz macierzy informacji prowadzi do znacznego uproszczenia procedur numerycznych.

SŁOWA KLUCZOWE: eksperyment płodozmianowy, układ dualny, układy $GCIB_2$, układ rozszerzony, współczynnik efektywności.